

On a Family of Exact Solutions to the Incompressible Liquid Crystals in Two Dimensions

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Abstract

In this paper we construct a family of exact strong solutions to the two-dimensional incompressible liquid crystal equations with finite energy. The initial velocity is chosen to be rotationally symmetric and the image of the initial orientation of the liquid crystal is a non-trivial curve on the unit sphere. It turns out that this family of initial data evolves globally in time by liquid crystal flow and may shrink to a single point as time goes to infinity.

1 Introduction

We consider the following hydrodynamic system modelling the flow of liquid crystal materials in two dimensions (see, for instance, [3, 6, 7]):

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \Delta \mathbf{u} - \nabla \cdot (\nabla \mathbf{d} \otimes \nabla \mathbf{d}), \\ \mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}, \\ \nabla \cdot \mathbf{u} = 0, \quad |\mathbf{d}| = 1, \end{cases} \quad (1.1)$$

where \mathbf{u} is the velocity field, p is the scalar pressure and \mathbf{d} is the unit-vector on the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ representing the macroscopic molecular orientation of the liquid crystal materials. Here the i th component of $\nabla \cdot (\nabla \mathbf{d} \otimes \nabla \mathbf{d})$ is given by $\nabla_j (\nabla_i \mathbf{d} \cdot \nabla_j \mathbf{d})$. For simplicity, we have set all the positive constants in the system to be one. We are interested in the Cauchy problem of (1.1) with the initial data

$$\mathbf{u}(0, x) = \mathbf{u}_0(x), \quad \mathbf{d}(0, x) = \mathbf{d}_0(x).$$

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The above system (1.1) is a simplified version of the Ericksen–Leslie model for the hydrodynamics of nematic liquid crystals [3, 6]. The mathematical analysis of the liquid crystal flow was initiated by Lin and Liu in [7, 8]. The existence of weak solutions in two dimensions is obtained in [10] where the authors also showed that there are at most finitely many time singularities for their weak solutions (see also [4]). The uniqueness of weak solutions in two dimensions was studied in [11, 13]. In [12], the global existence of strong solutions is proved for initial data with sufficiently small norms of \mathbf{u}_0 and $\nabla \mathbf{d}_0$ in BMO^{-1} . See also [9] for a small data global existence result in 3D. The global existence of large solutions is obtained in two dimensions in [2, 10] under the assumption that the image of \mathbf{d}_0 is contained in a half sphere (see also a recent new proof in [5]).

In this paper, we are concerned with a family of exact strong solutions with large initial data, which are global in time. We succeed in constructing them by choosing the rotationally symmetric initial velocity fields and symmetric initial orientations. Thus the image of the orientation must be a curve on \mathbb{S}^2 which could be non-trivial. We also show that the image curve of \mathbf{d} on \mathbb{S}^2 may finally shrink to a single point as time goes to infinity.

The main result of the paper is the following Theorem:

Theorem 1.1. *Let $\delta_1 \in (0, \pi/2)$ and u_0, ψ_0 be two given functions of $r \in \overline{\mathbb{R}^+}$ which satisfy*

$$\int_0^\infty u_0^2(r)r \, dr < \infty, \quad \int_0^\infty (\psi_0'(r))^2 r \, dr < \infty \quad \delta_1 < \psi_0(r) < \pi - \delta_1. \quad (1.2)$$

Then there exists a smooth function $\Phi : (\delta_1, \pi - \delta_1) \rightarrow \mathbb{R}$ such that, for the rotationally symmetric initial velocity field

$$\mathbf{u}_0(x) = u_0(|x|)e_\theta, \quad e_\theta = \left(-\frac{x_2}{|x|}, \frac{x_1}{|x|}\right)^T \quad (1.3)$$

and the symmetric initial orientation unit vector

$$\mathbf{d}_0(x) = \begin{pmatrix} \sin \psi_0(|x|) \cos \phi_0(|x|) \\ \sin \psi_0(|x|) \sin \phi_0(|x|) \\ \cos \psi_0(|x|) \end{pmatrix}, \quad \phi_0 = \Phi(\psi_0), \quad (1.4)$$

the solution to the liquid crystal equations (1.1) with initial data (1.3)-(1.4) is global and can be exactly solved.

Remark 1.1. The solution we obtained in Theorem 1.1 becomes smooth instantaneously for $t > 0$. By the weak-strong uniqueness (see, for instance, [11] and [13]), such solution is also unique.

We point out here that the L^2 norms of \mathbf{u}_0 and $\nabla \mathbf{d}_0$ may be large in our theorem. Hence the global well-posedness of such solutions is not a consequence of [12]. Since the image of \mathbf{d}_0 is not contained in any half sphere on \mathbb{S}^2 (cf. (3.6)), the existence of

our solutions cannot be deduced from [2, 10, 5] either. The main ingredient of this paper is that we can give an explicit expression of the solution to the liquid crystal equations (1.1) with initial data (1.3)-(1.4). For the proof, we first parameterize the curve, and solve a nonlinear ordinary differential equation (2.12) to reduce the problem to a nonlinear heat equation. We then construct a family of exact solutions by solving the nonlinear heat equation using a Hopf–Cole type transformation. We also show that the image curve of \mathbf{d} has non-increasing arc length along the liquid crystal flow. However, the arc length does not always decreasing in general. We will give an example of global solutions which does not shrink in time at the end of this paper.

Before ending this introduction, let us mention a related result on harmonic map heat flow in [1] where finite-time singularities are shown for a class of initial data in the form

$$\mathbf{d}_0(x) = \begin{pmatrix} x_1 r^{-1} \sin \psi_0(r) \\ x_2 r^{-1} \sin \psi_0(r) \\ \cos \psi_0(r) \end{pmatrix}$$

with

$$\psi_0(0) = 0, \quad \psi_0(R) > \pi \text{ for some } R > 0.$$

This also serves as an example of finite-time singularities for the liquid crystal system (1.1) in the special case when $\mathbf{u} \equiv 0$. Whether there are finite-time singularities for the incompressible liquid crystal flow in two dimensions with finite energy and non-trivial velocity remains a very interesting open question.

The paper is organized as follows: In Section 2 we will parameterize the orientation unit vector \mathbf{d} and rewrite the liquid crystal equations (1.1) in terms of ϕ and ψ . Then in Section 3 we solve (1.1) by giving an exactly expression of \mathbf{u} and \mathbf{d} , and finish the proof of Theorem 1.1.

2 Parameterization of the Liquid Crystal Equations

In this section we parameterize the liquid crystal equations. Let $e_r = x/|x|$. It is clear that the gradient operator can be expressed as

$$\nabla = e_r \partial_r + r^{-1} e_\theta \partial_\theta, \quad \partial_\theta = x_1 \partial_2 - x_2 \partial_1.$$

We look for solution of $(\mathbf{u}, \mathbf{d}, p)$, where \mathbf{u} is rotational and p, \mathbf{d} are symmetric, i.e.,

$$\mathbf{d} = \mathbf{d}(t, r), \quad \mathbf{u} = u(t, r) e_\theta, \quad p = p(t, r).$$

We can rewrite (1.1) into the following system

$$\begin{cases} u_t = u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u \\ p_r = \frac{1}{r} u^2 - \frac{1}{2} (|\mathbf{d}_r|^2)_r - \Delta \mathbf{d} \cdot \mathbf{d}_r \\ \mathbf{d}_t = \mathbf{d}_{rr} + \frac{1}{r} \mathbf{d}_r + |\mathbf{d}_r|^2 \mathbf{d} \end{cases} \quad (2.1)$$

Since $\mathbf{d} \in \mathbb{S}^2$, we may assume

$$\mathbf{d} = \begin{pmatrix} \sin \psi \cos \phi \\ \sin \psi \sin \phi \\ \cos \psi \end{pmatrix}. \quad (2.2)$$

A simple computation gives

$$\mathbf{d}_\psi = \begin{pmatrix} \cos \psi \cos \phi \\ \cos \psi \sin \phi \\ -\sin \psi \end{pmatrix}, \quad \mathbf{d}_\phi = \begin{pmatrix} -\sin \psi \sin \phi \\ \sin \psi \cos \phi \\ 0 \end{pmatrix}, \quad (2.3)$$

and

$$\mathbf{d}_{\psi\psi} = -\mathbf{d}, \quad \mathbf{d}_{\psi\phi} = \begin{pmatrix} -\cos \psi \sin \phi \\ \cos \psi \cos \phi \\ 0 \end{pmatrix}, \quad \mathbf{d}_{\phi\phi} = \begin{pmatrix} -\sin \psi \cos \phi \\ -\sin \psi \sin \phi \\ 0 \end{pmatrix}. \quad (2.4)$$

Clearly, from (2.2), (2.3), and (2.4), we have

$$\begin{aligned} \mathbf{d}_\psi \cdot \mathbf{d} &= 0, & \mathbf{d}_\psi \cdot \mathbf{d}_\psi &= 1, & \mathbf{d}_\psi \cdot \mathbf{d}_\phi &= 0, \\ \mathbf{d}_\psi \cdot \mathbf{d}_{\psi\psi} &= 0, & \mathbf{d}_\psi \cdot \mathbf{d}_{\psi\phi} &= 0, & \mathbf{d}_\psi \cdot \mathbf{d}_{\phi\phi} &= -\cos \psi \sin \psi, \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} \mathbf{d}_\phi \cdot \mathbf{d} &= 0, & \mathbf{d}_\phi \cdot \mathbf{d}_\psi &= 0, & \mathbf{d}_\phi \cdot \mathbf{d}_\phi &= \sin^2 \psi, \\ \mathbf{d}_\phi \cdot \mathbf{d}_{\psi\psi} &= 0, & \mathbf{d}_\phi \cdot \mathbf{d}_{\psi\phi} &= \sin \psi \cos \psi, & \mathbf{d}_\phi \cdot \mathbf{d}_{\phi\phi} &= 0. \end{aligned} \quad (2.6)$$

By the chain rule,

$$\begin{aligned} \mathbf{d}_t &= \mathbf{d}_\psi \psi_t + \mathbf{d}_\phi \phi_t, \\ \mathbf{d}_r &= \mathbf{d}_\psi \psi_r + \mathbf{d}_\phi \phi_r, \\ \mathbf{d}_{rr} &= \mathbf{d}_{\psi\psi} (\psi_r)^2 + 2\mathbf{d}_{\psi\phi} \psi_r \phi_r + \mathbf{d}_{\phi\phi} (\phi_r)^2 + \mathbf{d}_\psi \psi_{rr} + \mathbf{d}_\phi \phi_{rr}. \end{aligned}$$

Combining the above equalities with the third equation of (2.1), we obtain

$$\begin{aligned} \mathbf{d}_\psi \psi_t + \mathbf{d}_\phi \phi_t &= \mathbf{d}_{\psi\psi} (\psi_r)^2 + 2\mathbf{d}_{\psi\phi} \psi_r \phi_r + \mathbf{d}_{\phi\phi} (\phi_r)^2 + \mathbf{d}_\psi \psi_{rr} \\ &\quad + \mathbf{d}_\phi \phi_{rr} + \frac{1}{r} (\mathbf{d}_\psi \psi_r + \mathbf{d}_\phi \phi_r) + \mathbf{d} (\psi_r^2 + \sin^2 \psi \phi_r^2). \end{aligned} \quad (2.7)$$

We dot (2.7) with \mathbf{d}_ψ and \mathbf{d}_ϕ respectively and use (2.5) and (2.6) to deduce the following two equations

$$\psi_t = \psi_{rr} + \frac{1}{r} \psi_r - \cos \psi \sin \psi \phi_r^2, \quad (2.8)$$

$$\phi_t \sin^2 \psi = (\phi_{rr} + \frac{1}{r} \phi_r) \sin^2 \psi + 2 \cos \psi \sin \psi \psi_r \phi_r. \quad (2.9)$$

Note that in (2.7) the terms in the \mathbf{d} -direction cancel each other.

We assume that the image of \mathbf{d} is a curve on \mathbb{S}^2 . In other words, ϕ and ψ are not independent. Take $\phi = \Phi(\psi)$. Then it follows from (2.8) and (2.9) that

$$\psi_t = \psi_{rr} + \frac{1}{r}\psi_r - (\Phi')^2 \cos \psi \sin \psi \psi_r^2, \quad (2.10)$$

$$\Phi' \psi_t \sin^2 \psi = \Phi'(\psi_{rr} + \frac{1}{r}\psi_r) \sin^2 \psi + \Phi''(\psi_r)^2 \sin^2 \psi + 2\Phi' \cos \psi \sin \psi (\psi_r)^2.$$

By substituting the expression of ψ_t in (2.10) into the last equation, we then deduce

$$(\Phi')^3 \cos \psi \sin^3 \psi (\psi_r)^2 + \Phi'' \sin^2 \psi (\psi_r)^2 + 2\Phi' \cos \psi \sin \psi (\psi_r)^2 = 0. \quad (2.11)$$

A sufficient condition for (2.11) is the following ODE

$$\Phi'' \sin^2 \psi + 2\Phi' \cos \psi \sin \psi + (\Phi')^3 \cos \psi \sin^3 \psi = 0. \quad (2.12)$$

3 Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1 by explicitly solving the nonlinear ODE (2.12) and a nonlinear heat equation. Note that (2.12) is equivalent to

$$(\sin^2 \psi \Phi')' + \cos \psi \sin^3 \psi (\Phi')^3 = 0.$$

Thus, for $\psi \neq 0, \pi$, we have

$$\frac{(\sin^2 \psi \Phi')'}{(\sin^2 \psi \Phi')^3} = -\frac{\cos \psi}{\sin^3 \psi},$$

and then

$$\frac{1}{(\sin^2 \psi \Phi')^2} = -\frac{1}{\sin^2 \psi} + \beta$$

for some constant $\beta > 1$. Consequently,

$$(\Phi')^2 = \frac{1}{(\beta \sin^2 \psi - 1) \sin^2 \psi}. \quad (3.1)$$

Here we require

$$\beta \sin^2 \psi > 1. \quad (3.2)$$

Recall the condition on the initial data

$$\psi_0 \in (\delta_1, \pi - \delta_1). \quad (3.3)$$

We fix a $\beta \geq 1/\sin^2 \delta_1$. Plugging the expression (3.1) into (2.10), we get

$$\psi_t = \psi_{rr} + \frac{1}{r}\psi_r - \frac{\cos \psi}{(\beta \sin^2 \psi - 1) \sin \psi} \psi_r^2, \quad (3.4)$$

which is a nonlinear heat equation in 2D. This motivates us to use the idea of the Hopf–Cole transformation to reduce (3.4) to a linear equation. Let $F = F(\psi)$ be a function to be chosen. Clearly,

$$F_t = F'\psi_t, \quad F_{rr} + \frac{1}{r}F_r = F'(\psi_{rr} + \frac{1}{r}\psi_r) + F''\psi_r^2.$$

Then F satisfies the linear heat equation

$$F_t = F_{rr} + \frac{1}{r}F_r \tag{3.5}$$

provided that

$$F'' + F' \frac{\cos \psi}{(\beta \sin^2 \psi - 1) \sin \psi} = 0.$$

We solve the above ODE to get

$$F'(\psi) = C_1 \frac{\sin \psi}{(\beta \sin^2 \psi - 1)^{1/2}},$$

and

$$F(\psi) = \int C_1 \frac{\sin \psi}{(\beta \sin^2 \psi - 1)^{1/2}} d\psi = \frac{1}{\beta^{1/2}} \left(C_1 \arccos \left(\sqrt{\frac{\beta}{\beta - 1}} \cos \psi \right) + C_2 \right).$$

Here C_1 and C_2 are arbitrary constants. For our purpose, it is convenient to take $C_1 = \beta^{1/2}$ and $C_2 = 0$. Then

$$F(\psi) = \arccos \left(\sqrt{\frac{\beta}{\beta - 1}} \cos \psi \right).$$

Note that F is well defined for any $\psi \in (\delta_1, \pi - \delta_1)$, and it is a strictly increasing continuous function in the same interval. By (3.3), we have

$$F(\psi_0) \in (\delta_2, \pi - \delta_2), \quad \delta_2 = \arccos \left(\sqrt{\frac{\beta}{\beta - 1}} \cos \delta_1 \right) \in (0, \pi/2).$$

From (3.5), we have

$$F(\psi(t, \cdot)) = \Gamma(t, \cdot) * F(\psi_0),$$

where $\Gamma(t, \cdot)$ is the 2D heat kernel. Hence,

$$F(\psi(t, \cdot)) \in (\delta_2, \pi - \delta_2), \quad \psi(t, \cdot) \in (\delta_1, \pi - \delta_1),$$

which implies that the condition (3.2) is satisfied for any $t \geq 0$ if it is satisfied at $t = 0$. Therefore, the solution ψ exists globally. Furthermore, by the maximum principle, the length of the image curve of d is non-increasing.

It follows from (3.1) that

$$\Phi'(\psi) = \frac{1}{(\beta \sin^2 \psi - 1)^{1/2} \sin \psi}.$$

Here we took the positive branch of the square root. Therefore, by choosing a suitable constant,

$$\Phi(\psi) = \int_{\pi/2}^{\psi} \frac{1}{(\beta \sin^2 s - 1)^{1/2} \sin s} ds.$$

Observe that

$$\lim_{\psi \rightarrow \delta_1} \Phi(\psi) \rightarrow -\infty, \quad \lim_{\psi \rightarrow \pi - \delta_1} \Phi(\psi) \rightarrow +\infty \quad \text{as } \delta_1 \rightarrow 0. \quad (3.6)$$

We now treat the first two equations in (2.1). In the equation of u , we divide both sides by r and get

$$\left(\frac{u}{r}\right)_t = \left(\frac{u}{r}\right)_{rr} + \frac{3}{r} \left(\frac{u}{r}\right)_r. \quad (3.7)$$

We note that $\partial_{rr} + \frac{3}{r}\partial_r$ is the Laplace operator in \mathbb{R}^4 for radially symmetric functions. Moreover, as a radial function in \mathbb{R}^4 , $u_0/r \in L^2(\mathbb{R}^4)$ since, by (1.2),

$$\int_{\mathbb{R}^4} u_0^2(|y|)|y|^{-2} dy = C \int_0^\infty u_0^2(r)r dr < \infty.$$

Therefore, due to (3.7) we get

$$u(t, r) = r \int_{\mathbb{R}^4} \frac{1}{(4\pi t)^2} e^{-\frac{|x-y|^2}{4t}} u_0(t, |y|)|y|^{-1} dy, \quad x = r(1, 0, 0, 0),$$

and, as a radial function in \mathbb{R}^4 , $u(t, \cdot)/r \in L^2(\mathbb{R}^4)$ for any $t \geq 0$. This further implies that $\mathbf{u}(t, \cdot) \in L^2(\mathbb{R}^2)$. Since $\mathbf{u} = (-x_2, x_1)u/r$, we see that \mathbf{u} is a smooth function. Finally, from the second equation of (2.1), we get

$$p_r = \frac{1}{r}u^2 - \mathbf{d}_r \cdot \mathbf{d}_{rr} - (\mathbf{d}_{rr} + \frac{1}{r}\mathbf{d}_r) \cdot \mathbf{d}_r = \frac{1}{r}u^2 - 2\mathbf{d}_r \cdot \mathbf{d}_{rr} - \frac{1}{r}\mathbf{d}_r \cdot \mathbf{d}_r.$$

Therefore,

$$p(t, r) = \int_0^r \left(\frac{1}{s}u^2(t, s) - \frac{1}{s}|\mathbf{d}_r(t, s)|^2 \right) ds - |\mathbf{d}_r(t, r)|^2.$$

Note that by the radial symmetry condition, the integral above is convergence. This completes the proof of Theorem 1.1.

Remark 3.1. In the special case that the limit ψ_0 exists as $r \rightarrow \infty$, from the proof above it is easily seen that $\psi(t, \cdot)$ converges to this limit uniformly in \mathbb{R}^2 as t goes to infinity. Indeed, $F(\psi_0)$ is a bounded function and has a limit \bar{F} as $r \rightarrow \infty$. By the simple property of solutions to the heat equation, we have

$$\lim_{t \rightarrow \infty} F(\psi(t, \cdot)) \rightarrow \bar{F}$$

uniformly in \mathbb{R}^2 , which implies that

$$\lim_{t \rightarrow \infty} \psi(t, \cdot) \rightarrow \bar{\psi} := \arccos \left(\sqrt{\frac{\beta - 1}{\beta}} \cos \bar{F} \right)$$

uniformly in \mathbb{R}^2 . In this case, the image curve shrinks to a point on \mathbb{S}^2 uniformly along the liquid crystal flow.

On the other hand, one can find certain initial data such that the image curve on \mathbb{S}^2 does not shrink at all. This can be seen from the one-to-one correspondence of F and ψ and the following example. Let v_0 be a radial function in \mathbb{R}^2 defined as

$$v_0(r) = 0 \text{ on } [0, e], \quad v_0'(r) = \begin{cases} \frac{1}{r} \frac{1}{(6k+2)^3 - (6k+1)^3} & \text{on } (e^{(6k+1)^3}, e^{(6k+2)^3}] \\ 0 & \text{on } (e^{(6k+2)^3}, e^{(6k+4)^3}] \cup (e^{(6k+5)^3}, e^{(6k+7)^3}] \\ -\frac{1}{r} \frac{1}{(6k+5)^3 - (6k+4)^3} & \text{on } (e^{(6k+4)^3}, e^{(6k+5)^3}] \end{cases}, \quad k \in \mathbb{N}.$$

It is easy to check that $\nabla v_0 \in L^2(\mathbb{R}^2)$, $v_0 \in [0, 1]$, and

$$v_0(r) = \begin{cases} 1 & \text{on } [e^{(6k+2)^3}, e^{(6k+4)^3}] \\ 0 & \text{on } [e^{(6k+5)^3}, e^{(6k+7)^3}] \end{cases}, \quad k \in \mathbb{N}.$$

Now take $t_k = e^{2(6k+3)^3}$, $\tilde{t}_k = e^{2(6k+6)^3}$, and let v be the solution to the heat equation with initial data v_0 . Since

$$\lim_{k \rightarrow \infty} \frac{e^{2(6k+2)^3}}{t_k} = 0, \quad \lim_{k \rightarrow \infty} \frac{e^{2(6k+4)^3}}{t_k} = \infty,$$

we get

$$\lim_{k \rightarrow \infty} \int_{|x| \in (e^{(6k+2)^3}, e^{(6k+4)^3})} \Gamma(t_k, x) dx = 1.$$

Therefore,

$$\lim_{k \rightarrow \infty} v(t_k, 0) = 1.$$

Similarly, we have

$$\lim_{k \rightarrow \infty} v(\tilde{t}_k, 0) = 0.$$

Thus, the range of v does not shrink in time.

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